Announcements

1) Indexing error fixed on #2, HW 3

2) Midterm tentatively in two weeks (Monday)

Recall: (inner product)

If V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, an inner product is a function

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R} \text{ or } \mathbb{C}$$

such that

1) $\langle x, x \rangle \geq 0 \ \forall \ x \in V$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

2) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

$\forall\ x, y, z \in V$

3) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$

$\langle x, \alpha y\rangle = \bar{\alpha} \langle x, y\rangle$

$\forall\ x, y \in V,\ \alpha \in \mathbb{R}$ or $\mathbb{C}$

4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

In 2) and 3), if you know 4) as well, then you only need to check one of the properties listed in 2) and 3).

For example,

$$\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} \text{ by } 4)$$

$$= \overline{\alpha \langle y, x \rangle}$$

$$= \bar{\alpha}\, \overline{\langle y, x \rangle}$$

$$= \bar{\alpha} \langle x, y \rangle \text{ by } 4).$$

Example 1: ($\mathbb{C}^n$ or $\mathbb{R}^n$)

Let $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ (or $\mathbb{R}^n$)

and $w = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n$ (or $\mathbb{R}^n$).

Define

$$\boxed{\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}}$$

If these are real quantities, this is the dot product on $\mathbb{R}^n$.

Check this is an inner product

1) $\langle z, z \rangle = \sum_{i=1}^{n} z_i \overline{z}_i$

$$= \sum_{i=1}^{n} |z_i|^2$$

$$\geq 0$$

and is zero if and only if

$|z_i| = 0 \quad \forall \quad 1 \leq i \leq n,$

i.e. $z = (\underbrace{0, 0, \ldots, 0}_{n \text{ times}})$

4) $\langle z, w \rangle$

$$= \sum_{i=1}^{n} z_i \overline{w_i}$$

$$= \sum_{i=1}^{n} \overline{\overline{z_i} w_i}$$

$$= \overline{\sum_{i=1}^{n} \overline{z_i} w_i}$$

$$= \overline{\sum_{i=1}^{n} w_i \overline{z_i}}$$

$$= \overline{\langle w, z \rangle}$$

2) Let $v=(v_1, v_2, \cdots, v_n) \in \mathbb{C}^n$
$(\mathbb{R}^n)$.

Then

$$\langle z+w, v\rangle$$

$$= \sum_{i=1}^{n} (z_i+w_i)\overline{v_i}$$

$$= \sum_{i=1}^{n} \left(z_i\overline{v_i}+w_i\overline{v_i}\right)$$

$$= \sum_{i=1}^{n} z_i\overline{v_i}+\sum_{i=1}^{n} w_i\overline{v_i}$$

$$= \langle z, v\rangle + \langle w, v\rangle$$

3) Let α be a scalar.

Then

$$\langle \alpha z, w \rangle$$

$$= \sum_{i=1}^{n} (\alpha z_i) \overline{w_i}$$

$$= \alpha \sum_{i=1}^{n} z_i \overline{w_i}$$

$$= \alpha \langle z, w \rangle$$

So

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

gives an inner product on $\mathbb{C}^n$ (or $\mathbb{R}^n$).

Example 2: ($C(\mathbb{R})$ - kind of)

Let $f, g \in C(\mathbb{R})$.

Let $\int_{-\infty}^{\infty} f(x)\,dx$ denote

the (improper) Riemann

integral. Define

$$\boxed{\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx}$$

Check this is an inner product:

1) $\langle f, f \rangle = \int_{-\infty}^{\infty} (f(x))^2 dx$

≥ 0

If $\langle f,f \rangle = 0$ and

$\exists\ x \in \mathbb{R},\ f(x) \neq 0$, then

$\exists\ \varepsilon > 0$ such that

$f(x) \neq 0$ on $(x-\varepsilon, x+\varepsilon)$

by continuity of f.

Therefore we can write

$$\int_{-\infty}^{\infty} (f(x))^2 dx = \int_{-\infty}^{x-\varepsilon} (f(x))^2 dx + \int_{x-\varepsilon}^{x+\varepsilon} (f(x))^2 dx + \int_{x+\varepsilon}^{\infty} (f(x))^2 dx$$

Since $f(x) \neq 0$ on $(x-\varepsilon, x+\varepsilon)$,

$$\int_{x-\varepsilon}^{x+\varepsilon} (f(x))^2 \, dx > 0,$$

hence $\displaystyle\int_{-\infty}^{\infty} (f(x))^2 \, dx > 0$.

So $\langle f, f \rangle = 0$ if and only if $f(x) = 0$ for all x, i.e., f is the zero function.

4) $\langle f, g \rangle$

$$= \int_{-\infty}^{\infty} f(x) g(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$= \langle g, f \rangle = \overline{\langle g, f \rangle}$$

Since all numbers in this example are real.

2) Let $h \in C(\mathbb{R})$. Then

$$\langle f+g, h\rangle$$

$$= \int_{-\infty}^{\infty} (f+g)(x)\,h(x)\,dx$$

$$= \int_{-\infty}^{\infty} (f(x)+g(x))\,h(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(f(x)h(x)+g(x)h(x)\right)dx$$

$$= \int_{-\infty}^{\infty} f(x)h(x)\,dx + \int_{-\infty}^{\infty} g(x)h(x)\,dx$$

$$= \langle f, h\rangle + \langle g, h\rangle$$

3) Let $\alpha \in \mathbb{R}$. Then

$$\langle \alpha f, g \rangle$$

$$= \int_{-\infty}^{\infty} (\alpha f)(x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \alpha \cdot f(x) g(x)\, dx$$

$$= \alpha \int_{-\infty}^{\infty} f(x) g(x)\, dx$$

$$= \alpha \langle f, g \rangle$$

However, this is not an inner product on $C(\mathbb{R})$ since it could take the value ∞ or $-\infty$ (or just not exist!)

i.e. $f(x) = 1 \ \forall \ x \in \mathbb{R}$.

$$\langle f, f \rangle = \infty$$

If we restrict the inner product to all $f, g \in C(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} f(x)^2\,dx,\ \int_{-\infty}^{\infty} g(x)^2\,dx < \infty,$$

then this is an inner product!

Example 3: (matrices)

Let $(a_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{C})$

(or $M_n(\mathbb{R})$) considered

as a vector space over $\mathbb{C}$

(or over $\mathbb{R}$). Let

$A = (a_{i,j})_{i,j=1}^{n}$ and

define

$$A^* = (\overline{a_{j,i}})_{i,j=1}^{n}.$$

A^* is the adjoint of A and is equal to the transpose when $A \in M_n(\mathbb{R})$.

Define $\mathrm{Tr}: M_n(\mathbb{C}) \to \mathbb{C}$
$(M_n(\mathbb{R}) \to \mathbb{R})$

by $\mathrm{Tr}\left((a_{i,j})_{i,j=1}^{n}\right)$

$$= \sum_{i=1}^{n} a_{i,i} \quad ,$$ the

trace of the matrix.

For $A, B \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$),

define

$$\langle A, B \rangle = \operatorname{Tr}(B^* A).$$

This is an inner product!

To avoid excessive notation, what follows is the proof for $n=2$. Higher values of n are virtually identical in proof.

Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$

$B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$

and $C = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$

be elements of $M_2(\mathbb{C})$ (or $M_2(\mathbb{R})$).

Let α be a scalar.

1) $\langle A, A \rangle$

$$= \mathrm{Tr}(A^* A)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \overline{a}_{1,1} & \overline{a}_{2,1} \\ \overline{a}_{1,2} & \overline{a}_{2,2} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \right)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} |a_{1,1}|^2 + |a_{2,1}|^2 & \overline{a}_{1,1} a_{1,2} + \overline{a}_{2,1} a_{2,2} \\ \overline{a}_{1,2} a_{1,1} + \overline{a}_{2,2} a_{2,1} & |a_{1,2}|^2 + |a_{2,2}|^2 \end{pmatrix} \right)$$

$$= |a_{1,1}|^2 + |a_{2,1}|^2 + |a_{1,2}|^2 + |a_{2,2}|^2$$

$$\geq 0$$

Equality in the previous inequality only occurs when $|a_{i,j}| = 0$ for all $1 \leq i,j \leq 2$, which implies A is the zero matrix.

4) $\langle A, B \rangle$

$$= \operatorname{Tr}(B^* A)$$

$$= \operatorname{Tr}\left(\begin{pmatrix} \overline{b_{1,1}} & \overline{b_{2,1}} \\ \overline{b_{1,2}} & \overline{b_{2,2}} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \right)$$

$$= \operatorname{Tr}\left(\begin{pmatrix} \overline{b_{1,1}} a_{1,1} + \overline{b_{2,1}} a_{2,1} & \overline{b_{1,1}} a_{1,2} + \overline{b_{2,1}} a_{2,2} \\ \overline{b_{1,2}} a_{1,1} + \overline{b_{2,2}} a_{2,1} & \overline{b_{1,2}} a_{1,2} + \overline{b_{2,2}} a_{2,2} \end{pmatrix} \right)$$

$$= \overline{b_{1,1}} a_{1,1} + \overline{b_{2,1}} a_{2,1} + \overline{b_{1,2}} a_{1,2} + \overline{b_{2,2}} a_{2,2}$$

$$= a_{1,1} \overline{b_{1,1}} + a_{2,1} \overline{b_{2,1}} + a_{1,2} \overline{b_{1,2}} + a_{2,2} \overline{b_{2,2}}$$

$$\langle B, A \rangle = \mathrm{Tr}(A^{*}B)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \overline{a_{1,1}} & \overline{a_{2,1}} \\ \overline{a_{1,2}} & \overline{a_{2,2}} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \right)$$

$$= \mathrm{Tr}\left(\begin{pmatrix} \overline{a_{1,1}}\, b_{1,1} + \overline{a_{2,1}}\, b_{2,1} & \overline{a_{1,1}}\, b_{1,2} + \overline{a_{2,1}}\, b_{2,2} \\ \overline{a_{1,2}}\, b_{1,1} + \overline{a_{2,2}}\, b_{2,1} & \overline{a_{1,2}}\, b_{1,2} + \overline{a_{2,2}}\, b_{2,2} \end{pmatrix} \right)$$

$$= \overline{a_{1,1}}\, b_{1,1} + \overline{a_{2,1}}\, b_{2,1} + \overline{a_{1,2}}\, b_{1,2} + \overline{a_{2,2}}\, b_{2,2}$$

$$= \overline{a_{1,1}\, \overline{b_{1,1}} + a_{2,1}\, \overline{b_{2,1}} + a_{1,2}\, \overline{b_{1,2}} + a_{2,2}\, \overline{b_{2,2}}}$$

$$= \overline{\langle A, B \rangle}$$

2) $\langle A+B, C\rangle = \mathrm{Tr}(C^*(A+B))$

$$= \mathrm{Tr}(C^*A + C^*B)$$

$$\boxed{= \mathrm{Tr}(C^*A) + \mathrm{Tr}(C^*B)}$$

$$= \langle A, C\rangle + \langle B, C\rangle$$

Since for any matrices $S = (s_{i,j})_{i,j=1}^{2}$

and $T = (t_{i,j})_{i,j=1}^{2}$,

$$\mathrm{Tr}(T+S) = (t_{1,1}+s_{1,1}) + (t_{2,2}+s_{2,2})$$

$$= (t_{1,1}+t_{2,2}) + (s_{1,1}+s_{2,s})$$

$$= \mathrm{Tr}(T) + \mathrm{Tr}(S)$$

3) $\langle \alpha A, B \rangle$

$$= \mathrm{Tr}(B^*(\alpha A))$$

$$= \mathrm{Tr}(\alpha B^* A)$$

$$\boxed{= \alpha \,\mathrm{Tr}(B^* A)} = \alpha \langle A, B \rangle$$

Since for any $T = (t_{i,j})_{i,j=1}^{2}$

$$\mathrm{Tr}(\alpha T) = \alpha t_{1,1} + \alpha t_{2,2}$$

$$= \alpha (t_{1,1} + t_{2,2})$$

$$= \alpha \,\mathrm{Tr}(T).$$

Therefore

$$\langle A, B \rangle = \operatorname{Tr}(B^* A)$$

is an inner product on $M_2(\mathbb{C})$ (or $M_n(\mathbb{R})$).